Remarks about realizability

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PROLOGUE

The full internal category on modest sets in the effective topos is sufficiently complete to provide an elementary model of polymorphism, $i.e.$ one where type abstraction is interpreted as products over the collection of all types provided the semanticist is prepared to use intuitionistic set theory in place of standard set theory. But that internal category is not complete.

Thus Peter Freyd's remark that an internal complete category of a (Grothendieck) topos is posetal could still hold for any topos—without the parenthesized assumption.

We show that the 1-category reflection of the 2-category $\mathbb{PGRPD}$ of the effective topos consisting of those internal $\mathbb{PER}$-enriched groupoids with a projective object of objects, with enriched functors and natural transformations, gives a topos with an internal non-posetal category which is complete.

Since it is complete it provides, in particular, another elementary model of polymorphism, and we believe that that is also parametric.

OVERVIEW

THEOREM. There is an elementary topos $\mathbb{G}$ with an internal non-posetal complete category.

REMARKS.

1. The topos $\mathbb{G}$ is the 1-category reflection of the 2-category $\mathbb{PGRPD}$ of the effective topos on those internal $\mathbb{PER}$-enriched groupoids with a projective object of objects, with functors and natural transformations.

2. The internal complete category is essentially given by the groupoid of isomorphisms between PER's.

"The Logic of Proofs": a primer for the Effective Topos

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$$n \in P \wedge Q \overset{\text{df}}{\Leftrightarrow} n = \langle n_0, n_1 \rangle \text{ and } n_0 \in P, n_1 \in Q$$

$$n \in P \vee Q \overset{\text{df}}{\Leftrightarrow} n = \langle n_0, n_1 \rangle; n_0 = 0, 1; \text{ if } n_0 = 0\text{, then } n_1 \in P\text{; if } n_0 = 1\text{, then } n_1 \in Q$$

$$n \in P \Rightarrow Q \overset{\text{df}}{\Leftrightarrow} \text{for all } k \in P, \varphi_n(k)\downarrow \text{ and } \varphi_n(k) \in Q$$

$$n \in \forall_i\ P_i \overset{\text{df}}{\Leftrightarrow} \text{for all } i,\ n \in P_i$$

$$n \in \exists_i\ P_i \overset{\text{df}}{\Leftrightarrow} \text{for some } i,\ n \in P_i$$

$$n \in \top \overset{\text{df}}{\Leftrightarrow} \text{always}$$

$$n \in \bot \overset{\text{df}}{\Leftrightarrow} \text{never}$$

The Effective Topos

$X = (|X|, \mathbb{X}\colon |X| \times |X| \to \mathbf{P}(\mathbb{N}))$ and

there are numbers s and c such that

$$\varphi_s(\mathbb{X}(x_1, x_2)) \subseteq \mathbb{X}(x_2, x_1) \qquad \text{for all } x_1, x_2 \in |X|$$
$$\varphi_c(\mathbb{X}(x_1, x_2) \wedge \mathbb{X}(x_2, x_3)) \subseteq \mathbb{X}(x_1, x_3) \qquad \text{for all } x_1, x_2, x_3 \in |X|$$

$X \xrightarrow{[F, f]} Y$ is an equivalence class of pairs consisting of

a function F and a number f such that

$$F\colon \{(x, a) \mid a \in \mathbb{X}(x, x)\} \to |Y|$$
$$\varphi_{\varphi_f(a_1, a_2)}(\mathbb{X}(x_1, x_2)) \subseteq \mathbb{Y}(F(x_1, a_1), F(x_2, a_2)) \qquad \text{for all } x_1, x_2, a_1, a_2$$

$$\begin{tikzcd} X \arrow[r, bend left, "{(F, f)}"] \arrow[r, bend right, "{(G, g)}"'] \arrow[r, phantom, "\wr"] & Y \end{tikzcd}$$ when

there is a number e such that

$$\varphi_{\varphi_e(a_1)}(\mathbb{X}(x_1, x_1)) \subseteq \mathbb{Y}(F(x_1, a_1), G(x_1, a_1)) \qquad \text{for all } x_1, a_1$$

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A complicated presentation of the Effective Topos

$X = (|X|, \mathbb{X} \colon |X| \times |X| \to \mathrm{PER}_{\mathrm{t/d}})$ and

there are numbers s and c such that

$$\varphi_s \colon \mathbb{X}(x_1, x_2) \to \mathbb{X}(x_2, x_1)$$
$$\varphi_c \colon \mathbb{X}(x_1, x_2) \times \mathbb{X}(x_2, x_3) \to \mathbb{X}(x_1, x_3)$$

$X \xrightarrow{[F,f]} Y$ is an equivalence class of pairs consisting of

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$$(F, \varphi_{\varphi_f \langle \pi_2, \pi_2 \rangle}) \colon \{(x, a) \mid a \in \mathbb{X}(x, x)\} \to \{ (y, b) \mid b \in \mathbb{Y}(y, y)\}$$
$$\varphi_{\varphi_f(a_1, a_2)} \colon \mathbb{X}(x_1, x_2) \to \mathbb{Y}(F(x_1, a_1), F(x_2, a_2))$$

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$X = (|X|, \mathbb{X}\colon |X| \times |X| \to \mathrm{PER}_{\mathrm{t/d}})$ and

there are numbers s and c such that

$\mathrm{t/d}$ stand for "total on the domain"
i.e. $P = \mathrm{dom}(P) \times \mathrm{dom}(P)$.

$$\begin{aligned}\varphi_s &\colon \mathbb{X}(x_1, x_2) \to \mathbb{X}(x_2, x_1)\\ \varphi_c &\colon \mathbb{X}(x_1, x_2) \times \mathbb{X}(x_2, x_3) \to \mathbb{X}(x_1, x_3)\end{aligned}$$

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An even more complicated presentation

$$X = \left(\begin{array}{c} ||X|| \\ \pi \big\downarrow \\ \mathbb{N} \end{array}, \mathbb{X}\colon ||X|| \times ||X|| \to \mathrm{PER}_{\mathrm{t/d}} \right)$$ and

numbers i, c, and s such that

$$\begin{array}{l} \varphi_i(\pi(\overline{x})) \in \mathbb{X}(\overline{x}, \overline{x}) \\ \varphi_{\varphi_c(\pi(\overline{x}_1), \pi(\overline{x}_2))}\colon \mathbb{X}(\overline{x}_1, \overline{x}_2) \times \mathbb{X}(\overline{x}_2, \overline{x}_3) \to \mathbb{X}(\overline{x}_1, \overline{x}_3) \\ \varphi_{\varphi_s(\pi(\overline{x}_1), \pi(\overline{x}_2))}\colon \mathbb{X}(\overline{x}_1, \overline{x}_2) \to \mathbb{X}(\overline{x}_2, \overline{x}_1) \end{array}$$

$X \xrightarrow{[\overline{F}, n, f]} Y$ is an equivalence class of triples consisting of

a function $\overline{F}$ and two numbers n and f such that

$$\overline{F}\colon ||X|| \to ||Y|| \qquad \text{with } \sigma \circ \overline{F} = \varphi_n \circ \pi$$

$$\varphi_{\varphi_f(\pi(\overline{x}_1), \pi(\overline{x}_2))}\colon \mathbb{X}(\overline{x}_1, \overline{x}_2) \to \mathbb{Y}(\overline{F}(\overline{x}_1), \overline{F}(\overline{x}_2))$$

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$$||X|| \stackrel{\text{df}}{=} \{(x,a) \mid x \in |X|, a \in \mathbb{X}(x,x)\}$$

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$\|X\| \stackrel{\mathsf{df}}{=} \{(x,a) \mid x \in |X|, a \in \mathbb{X}(x,x)\}$

$\mathbb{X}((x_1,a_1),(x_2,a_2)) \stackrel{\mathsf{df}}{=} \mathbb{X}(x_1,x_2)\}$

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A 2-category whose 1-reflection is the Effective Topos

$$X = \left(\begin{array}{c} ||X|| \\ \pi\big\downarrow \\ \mathbb{N} \end{array}, \mathbb{X}\colon ||X|| \times ||X|| \to \mathrm{PER}_{\mathrm{t/d}} \right)$$ and

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$$\varphi_{\varphi_s(\pi(\overline{x}_1),\pi(\overline{x}_2))}\colon \mathbb{X}(\overline{x}_1, \overline{x}_2) \to \mathbb{X}(\overline{x}_2, \overline{x}_1)$$

$X \xrightarrow{(\overline{F}, \varphi_n, \varphi_f)} Y$ is a triple consisting of

three functions $\overline{F}$, φ_n and φ_f such that

$$\overline{F}\colon ||X|| \to ||Y|| \qquad \text{with } \sigma \circ \overline{F} = \varphi_n \circ \pi$$
$$\varphi_{\varphi_f(\pi(\overline{x}_1),\pi(\overline{x}_2))}\colon \mathbb{X}(\overline{x}_1, \overline{x}_2) \to \mathbb{Y}(\overline{F}(\overline{x}_1), \overline{F}(\overline{x}_2))$$

$$\begin{tikzcd} X \arrow[r, bend left, "(\overline{F}, \varphi_n, \varphi_f)"] \arrow[r, bend right, "(\overline{G}, \varphi_\ell, \varphi_g)"'] \arrow[r, phantom, "{[e]\downarrow\cdot}"] & Y \end{tikzcd}$$ when

$$\varphi_e(\pi(\overline{x})) \in \mathbb{Y}(\overline{F}(\overline{x}), \overline{G}(\overline{x}))$$

and any two such numbers are equivalent.

The 2-category $\mathbb{PGRPD}$ of groupoids whose 1-reflection is $\mathbb{G}$

$$X = \left(\begin{array}{c} ||X|| \\ \pi\big\downarrow \\ \mathbb{N} \end{array}, \mathbb{X}\colon ||X|| \times ||X|| \to \mathrm{PER} \right)$$ and

numbers i, c, and s such that

$$\begin{aligned} &\varphi_i(\pi(x)) \in \mathbb{X}(x,x) \\ &\varphi_{\varphi_c(\pi(x_1),\pi(x_2))}\colon \mathbb{X}(x_1,x_2) \times \mathbb{X}(x_2,x_3) \to \mathbb{X}(x_1,x_3) \\ &\varphi_{\varphi_s(\pi(x_1),\pi(x_2))}\colon \mathbb{X}(x_1,x_2) \to \mathbb{X}(x_2,x_1) \end{aligned}$$

form a $\mathbb{PER}$-groupoid.

$X \xrightarrow{(F,\varphi_n,\varphi_f)} Y$ is a triple consisting of

three functions F, φ_n and φ_f such that

$$F\colon ||X|| \to ||Y|| \qquad \text{with } \sigma \circ F = \varphi_n \circ \pi$$

$$\varphi_{\varphi_f(\pi(x_1),\pi(x_2))}\colon \mathbb{X}(x_1,x_2) \to \mathbb{Y}(F(x_1),F(x_2))$$

form an enriched functor.

$$\begin{tikzcd} X \arrow[r, bend left, "{(F,\varphi_n,\varphi_f)}"] \arrow[r, bend right, "{(G,\varphi_\ell,\varphi_g)}"'] \arrow[r, phantom, "{[e]\downarrow\cdot}"] & Y \end{tikzcd}$$ when

$$\varphi_e(\pi(x)) \in \mathbb{Y}(F(x),G(x))$$

is an enriched nat.transf.,

and any two such numbers are equivalent if $\varphi_{e_1}(\pi(x))\ \mathbb{Y}(F(x),G(x))\ \varphi_{e_2}(\pi(x))$.

Properties of the 2-category $\mathbb{PGRPD}$

- It has 2-products

- It has weak inserters

- It has weak exponentiation

- It has a 2-factorization system given by surjective-on-objects functors and full embeddings

- In the 1-reflection, a monomorphism $X \xrightarrow{(M, \varphi_j, \varphi_m)} Y$ is (isomorphic to) a full embedding

  since the diagonal from X into the pullback P of the map M with itself is the inverse of either projection from the pullback, up to isomorphism. The natural iso from the identity on the pullback to the composition of the first projection and the diagonal yields fullness.

Properties of the 2-category $\mathbb{PGRPD}$

- It has a 2-factorization system given by surjective-on-objects functors and full embeddings

- So $\mathrm{true}\colon 1 \rightarrowtail \Omega$ is a subobject classifier in the 1-reflection $\mathbb{G}$

- $\mathbb{G}$ is a topos

- The object of objects of the complete category is the set PER of all per's with a constant function together with the map $\mathrm{iso}\colon \mathrm{PER} \times \mathrm{PER} \longrightarrow \mathrm{PER}$.

- The object of maps is the set of triples (A, p, B) such that $\varphi_p\colon A \longrightarrow B$ is a map from the per A to the per B with second projection into $\mathbb{N}$, together with the map that sends a pair of such triples $((A_1, p_1, B_1), (A_2, p_2, B_2))$ to the per of all "commutative squares" (a, b) on them
$$\begin{tikzcd} A_1 \arrow[r, "\varphi_a"] \arrow[d, "\varphi_{p_1}"'] & A_2 \arrow[d, "\varphi_{p_2}"] \\ B_1 \arrow[r, "\varphi_b"'] & B_2 \end{tikzcd}$$

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