Fix Your Semantic Cube Using This Simple Trick

Pierre Clairambault
CNRS, LIP, ENS Lyon

\[
P \parallel Q
\]

\[
\lambda \quad \text{rand()}
\]

\[
x := \text{tt}
\]

Bath (sort of), 14/04/20.
I. BACKGROUND : GAME SEMANTICS
Game Semantics by Example (Call-By-Name)

A term:

\[ \lambda f^{U \to U}. \text{newref } r \text{ in } f (r := \text{tt}); !r : (U \to U) \to B \]
A term:

$$\lambda f^{\mathbb{U} \rightarrow \mathbb{U}}. \text{newref } r \text{ in } f \,(r := \text{tt}); \,!r : (\mathbb{U} \rightarrow \mathbb{U}) \rightarrow \mathbb{B}$$

A game:

```
A game

q^-

q^+

q^+

(\_\_\_\_)

(\_\_\_\_)

\text{tt}^+

\text{ff}^+

\text{ff}^+

\text{tt}^+

(\_\_\_\_)

(\_\_\_\_)

```

The strategy interpreting a term is the set of plays realized by that term.
A **term**:

$$\lambda f : U \rightarrow U. \text{newref } r \text{ in } f \ (r := \texttt{tt}); \ !r : \ (U \rightarrow U) \rightarrow B$$

A **game**

```
A game

q^- q^+ q^- q^- ()^- ()^- ()^+ tt^+ ff^+
```
A term:

\[ \lambda f^{U \to U}. \text{newref } r \text{ in } f \ (r := \text{tt}); \ !r : (U \to U) \to B \]
A term:

\[ \lambda f^{U \to U}. \text{newref } r \text{ in } f (r := \texttt{tt}); !r : (U \to U) \to \mathbb{B} \]
Game Semantics by Example (Call-By-Name)

A term:

\[ \lambda f : U \rightarrow U. \text{newref } r \text{ in } f (r := \texttt{tt}); \,!r \quad : \quad (U \rightarrow U) \rightarrow \mathbb{B} \]

A game

\[ \begin{aligned}
q^- & \quad q^+ \\
\_ & \quad \text{tt}^+ \sim \text{ff}^+ \\
q^- & \quad ()^- \\
\_ & \quad ()^+ 
\end{aligned} \]
Game Semantics by Example (Call-By-Name)

A term:

$$\lambda f : U \rightarrow U. \text{newref } r \text{ in } f (r := \text{tt}); !r : (U \rightarrow U) \rightarrow B$$

A play

$$(U \rightarrow U) \rightarrow B$$

A game

$$q^-$$

$$q^+$$

$$\text{tt}^+ \sim \text{ff}^+$$

$$(\cdot)^-$$

$$(\cdot)^+$$
Game Semantics by Example (Call-By-Name)

A term:

\[ \lambda f : U \to U. \, \text{newref } r \text{ in } f \,(r := \text{tt}); \, !r \quad : \quad (U \to U) \to \mathbb{B} \]

A play

\[
(U \to U) \to \mathbb{B}
\]

A game

\[
\begin{array}{c}
q^- \\
q^+ \\
\text{tt}^+ \sim \text{ff}^+ \\
q^- \\
()^-
\end{array}
\]
A term:

$$\lambda f : U \to U. \mathsf{newref} \ r \ \mathsf{in} \ f \ (r := tt) ; ! r : (U \to U) \to B$$
Game Semantics by Example (Call-By-Name)

A term:

$$\lambda f^{U \to U}. \text{newref } r \text{ in } f \ (r := \text{tt}); \ !r \ : \ (U \to U) \to B$$

A play

$$ (U \to U) \to B$$

A game

$$q^- \quad q^+ \quad \texttt{tt}^+ \sim \texttt{ff}^+$$

The strategy interpreting a term is the set of plays realized by that term.
A **term**:

$$\lambda f^{U \rightarrow U}. \text{newref } r \text{ in } f \ (r := \text{tt}); \ !r \ : \ (U \rightarrow U) \rightarrow B$$

A **play**:

$$\begin{align*}
&\quad \quad (U \rightarrow U) \rightarrow B \\
&\quad \quad \quad \quad \quad q^- \\
&\quad \quad \quad \quad q^+ \\
&\quad \quad q^- \\
&()^+
\end{align*}$$

A **game**:

$$\begin{align*}
&\quad \quad \quad \quad q^+ \\
&\quad \quad \quad \quad tt^+ \sim ff^+ \\
&\quad \quad \quad \quad q^- \\
&\quad \quad \quad ()^- \\
&()^+
\end{align*}$$

The strategy interpreting a term is the set of plays realized by that term.
A term:

$$\lambda f^{U \to U}. \text{newref } r \text{ in } f (r := \texttt{tt}); \ !r : (U \to U) \to B$$

A play

\[
\begin{array}{ccc}
(U \to U) & \to & B \\
    & \q^+ & \q^- \\
    \q^- & \text{()^+} & \text{()}^- \\
\end{array}
\]

A game

\[
\begin{array}{cccc}
\text{q^-} & \text{tt^+} & \sim & \text{ff^+} \\
\text{q^+} & \text{q^-} & \text{()}^- & \text{()}^+ \\
\end{array}
\]
A term:

\[ \lambda f : U \rightarrow U. \text{newref } r \text{ in } f (r := \text{tt}); \!r \quad : \quad (U \rightarrow U) \rightarrow \mathbb{B} \]

A play

\[
\begin{array}{c}
(U \rightarrow U) \rightarrow \mathbb{B} \\
q^- \\
q^+ \\
q^- \\
()^+ \\
()^- \\
()^-
\end{array}
\]

A game

\[
\begin{array}{c}
q^- \\
q^+ \\
\text{tt}^+ \sim \text{ff}^+ \\
q^- \\
()^- \\
()^+
\end{array}
\]
A term:

$$\lambda f : U \rightarrow U. \text{newref } r \in f (r := \text{tt}); !r : (U \rightarrow U) \rightarrow B$$

A play:

$$(U \rightarrow U) \rightarrow B$$

A game:

$$(\text{tt}^+) \sim (\text{ff}^+)$$

The strategy interpreting a term is the set of plays realized by that term.
A term:

$$\lambda f : \mathbb{U} \rightarrow \mathbb{U}. \text{newref } r \text{ in } f (r := \text{tt}); \ !r : (\mathbb{U} \rightarrow \mathbb{U}) \rightarrow \mathbb{B}$$

A play:

$$(\mathbb{U} \rightarrow \mathbb{U}) \rightarrow \mathbb{B}$$

A game:

$$q^- \quad q^+ \quad tt^+ \sim ff^+$$

The strategy interpreting a term is the set of plays realized by that term.
A term:

\[ \lambda f^{U \to U}. \text{newref } r \text{ in } f (r := \text{tt}); \!r \ : \ (U \to U) \to B \]
A term:

\[ \lambda f : U \to U. \text{newref } r \text{ in } f (r := \text{tt}); !r : (U \to U) \to B \]

A play

\[
\begin{array}{c}
(U \to U) \to B \\
q^- \\
q^+ \\
()^-
\end{array}
\]

A game

\[
\begin{array}{c}
q^- \\
q^+ \\
tt^+ \sim ff^+ \\
()^-
\end{array}
\]
Game Semantics by Example (Call-By-Name)

A term:

\[ \lambda f : U \rightarrow U. \text{newref } r \ in \ f \ (r := \text{tt}); \ !r : (U \rightarrow U) \rightarrow \mathbb{B} \]

A play

\[(U \rightarrow U) \rightarrow \mathbb{B} \]

\[q^- \]

\[q^+ \]

\[()^- \]

\[\text{ff}^+ \]

A game

\[q^- \]

\[q^+ \]

\[\text{tt}^+ \sim \text{ff}^+ \]

\[q^- \]

\[()^- \]

\[()^+ \]

The strategy interpreting a term is the set of plays realized by that term.
Types as Games as Event Structures

Definition

An **event structure** is a tuple \( E = \langle |E|, \leq_E, \#_E \rangle \) where:

- \(|E|\) is a set of **events**,
- \(\leq_E\) is a partial order called **causality**,
- \(\#_E\) is an irreflexive symmetric binary relation called **conflict**.

satisfying some axioms. A **game** is an event structure \( A \) with

\[
\text{pol}_A : |A| \rightarrow \{-, +\}
\]

indicating for each event its **polarity**.

Games as Event Structures

\[
(U \rightarrow U) \rightarrow B
\]
Definitions

A (finite) configuration of an event structure $E$ is a finite set $x \subseteq |E|$ which is:

- **Down-closed**: for all $e \in x$, for all $e' \leq_E e$, we have $e' \in x$;
- **Consistent**: for all $e, e' \in x$, we have $\neg (e \#_E e')$.

The set of (finite) configurations of $E$ is written $C(E)$.
Plays

Definition

An **alternating play** on game $A$ is a finite non-repetitive sequence of events $a_1 \ldots a_n$ such that $\text{pol}_A(a_1) = -$, for all $1 \leq i \leq n$, $\text{pol}_A(a_i) \neq \text{pol}_A(a_{i+1})$ and

$\{a_1, \ldots, a_i\} \in C(A)$.

We write $\text{AltPlays}(A)$ the set of (alternating) plays on $A$.
**Definition**

An *(alternating) play* on game $A$ is a finite non-repetitive sequence of events $a_1 \ldots a_n$ such that $\text{pol}_A(a_1) = -, \text{ for all } 1 \leq i \leq n, \text{ pol}_A(a_i) \neq \text{ pol}_A(a_{i+1})$ and 

$$\{a_1, \ldots, a_i\} \in C(A).$$

We write $\text{AltPlays}(A)$ the set of (alternating) plays on $A.$
Plays

Definition

An (alternating) play on game $A$ is a finite non-repetitive sequence of events $a_1 \ldots a_n$ such that $\text{pol}_A(a_1) = -1$, for all $1 \leq i \leq n$, $\text{pol}_A(a_i) \neq \text{pol}_A(a_{i+1})$ and

$$\{a_1, \ldots, a_i\} \in C(A).$$

We write $\text{AltPlays}(A)$ the set of (alternating) plays on $A$. 

A play

$$(U \rightarrow U) \rightarrow B$$

A game

```
\begin{align*}
q^- & \quad q^+ \\
  &  \quad t^+ \sim f^+ \\
  &  \quad ()^- \\
  &  \quad ()^+ \\
\end{align*}
```
Plays

Definition

An **alternating** play on game $A$ is a finite non-repetitive sequence of events $a_1 \ldots a_n$ such that $\text{pol}_A(a_1) = -$, for all $1 \leq i \leq n$, $\text{pol}_A(a_i) \neq \text{pol}_A(a_{i+1})$ and

$\{a_1, \ldots, a_i\} \in C(A)$.

We write $\text{AltPlays}(A)$ the set of (alternating) plays on $A$. 

A **play**

$$(U \rightarrow U) \rightarrow B$$

A **game**

$$q^- \sim \sim \sim tt^+ \rightarrow ff^+$$

$$q^- \rightarrow (\cdot^-)$$

$$\cdot^+$$
Plays

Definition

An (alternating) play on game $A$ is a finite non-repetitive sequence of events $a_1 \ldots a_n$ such that $\text{pol}_A(a_1) = -$, for all $1 \leq i \leq n$, $\text{pol}_A(a_i) \neq \text{pol}_A(a_{i+1})$ and $
{a_1, \ldots, a_i} \in C(A)$.

We write $\text{AltPlays}(A)$ the set of (alternating) plays on $A$. 

A play

$$(U \rightarrow U) \rightarrow B$$

A game

$$q^- \quad q^+$$

$$q^- \quad (\_)^+$$

$$q^- \quad (\_)^-$$
**Plays**

**Definition**

An *alternating* play on game $A$ is a finite non-repetitive sequence of events $a_1 \ldots a_n$ such that $\pol_A(a_1) = -$ for all $1 \leq i \leq n$, $\pol_A(a_i) \neq \pol_A(a_{i+1})$ and

$$\{a_1, \ldots, a_i\} \in C(A).$$

We write $\textbf{AltPlays}(A)$ the set of (alternating) plays on $A$. 

---

**A play**

$$
\begin{align*}
(U \rightarrow U) \rightarrow B \\
\uparrow & \\
q^+ & \\
q^- \\
(\cdot)^+ & \\
(\cdot)^- \\
\end{align*}
$$

**A game**

$$
\begin{align*}
\vdots & \\
q^- \\
(\cdot)^- & \\
\vdots \\
(q^-)^+ & \\
(\cdot)^+ & \\
\end{align*}
$$
Plays

Definition

An **(alternating) play** on game $A$ is a finite non-repetitive sequence of events $a_1 \ldots a_n$ such that $\text{pol}_A(a_1) = -$, for all $1 \leq i \leq n$, $\text{pol}_A(a_i) \neq \text{pol}_A(a_{i+1})$ and

$$\{a_1, \ldots, a_i\} \in C(A).$$

We write $\text{AltPlays}(A)$ the set of (alternating) plays on $A$. 
The game for \( (U \rightarrow U) \rightarrow B \) with repetitions
The game for \((U \to U) \to B\) with repetitions

A play with repetitions

\[(U \to U) \to B\]
Question: is this play realisable?

\[(U \rightarrow U) \rightarrow B\]

This play is non well-bracketed and cannot be realised without callcc.
Question: is this play realisable?

\((\mathbb{U} \to \mathbb{U}) \to \mathbb{B}\)

\(\lambda f^{\mathbb{U} \to \mathbb{U}} \cdot \text{callcc} (\lambda k^{\mathbb{B} \to \mathbb{U}} \cdot f(k \mathbb{tt})) : (\mathbb{U} \to \mathbb{U}) \to \mathbb{B}\)
Question: is this play realisable?

\[ (U \to U) \to B \]

\[ \lambda f^{U \to U}. \text{callcc} (\lambda k^{B \to U}. f (k \text{tt})) : (U \to U) \to B \]
Question: is this play realisable?

\[ (\mathbb{U} \rightarrow \mathbb{U}) \rightarrow \mathbb{B} \]

\[ \rightarrow q^-, Q \]

\[ q^-, Q \rightarrow q^+, Q \rightarrow \mathbb{tt}^+, A \]

\[ \lambda f^{U \rightarrow U} \cdot \text{callcc} (\lambda k^{B \rightarrow U} \cdot f (k \mathbb{tt})) : (U \rightarrow U) \rightarrow \mathbb{B} \]

**Theorem**

This play is **non well-bracketed** and cannot be realized without **callcc**.
Question: is this play realisable?

\[(\mathbb{B} \to U) \to U\]

Theorem

This play is non-innocent and cannot be realised without references.
Question: is this play realisable?

\[ (B \to U) \to U \]

\[ \lambda f^{B \to U}. \text{newref } r \text{ in } f \text{ (let } x = !r \text{ in } r := \text{tt}; x) : (B \to U) \to U \]
Question: is this play realisable?

\[(\mathbb{B} \rightarrow \mathbb{U}) \rightarrow \mathbb{U}\]

\[
\lambda f^{\mathbb{B}\rightarrow \mathbb{U}} . \text{newref } r \text{ in } f \ (\text{let } x =! r \text{ in } r := tt; \ x) : (\mathbb{B} \rightarrow \mathbb{U}) \rightarrow \mathbb{U}
\]

**Theorem**

This play is **non-innocent** and cannot be realized without references.
Full abstraction results

all correspondences being **fully abstract** or **intensionally fully abstract**.¹

¹Follows from work in the late 90s from Abramsky, Hyland, Laird, McCusker, Ong.
Orthogonality of control and state

Theorem

Suppose a program \( M \) in \( \text{cIA} \) is observationally equivalent to

- A program \( M_1 \) that does not use \text{callcc};
- A program \( M_2 \) that does not use \text{references}.

Then, \( M \) is observationally equivalent to \( M' \) in pure \( \text{PCF} \).
The “semantic cube”
The “semantic cube”
The “semantic cube”
The "semantic cube"

IA + parallelism
GM

cPCF
Strat\textsubscript{inn} <-> cIA
Strat

Strat\textsubscript{inn,wb} <-> Strat\textsubscript{wb}

PCF
IA
The “semantic cube”
The “semantic cube”
The “semantic cube”
Outline

PCF

PPCF

+parallelism

+probabilities

+state

IPA

PIA

PART II

PART III

PPCF||

PIPA
II. Concurrent Games and Parallel Innocence
IPA and its components

Types.

\[ A, B ::= U \mid B \mid N \mid A \to B \quad \text{PCF} \]
\[ \mid \text{ref} \quad +\text{state} \]

Terms.

\[ M, N ::= x \mid M \, N \mid \lambda x. \, M \mid Y \quad \lambda Y\text{-calculus} \]
\[ \mid \text{tt} \mid \text{ff} \mid \text{if} \, M \, N_1 \, N_2 \]
\[ \mid n \mid \text{succ} \, M \mid \text{pred} \, M \mid \text{iszero} \, M \]
\[ \mid \text{skip} \mid M; N \quad \text{PCF} \]
\[ \mid \text{newref} \, v := b \, \text{in} \, M \mid M := N \mid !M \quad +\text{state} \]
\[ \mid \text{let} \left( \begin{array}{c}
  x = M \\
  y = N
\end{array} \right) \text{in} \, T \quad +\text{parallel} \]

\[ \iff \text{PCF} + \text{state} + \text{parallel} = \text{IPA} \]

Standard typing rules and call-by-name operational semantics.
Roadmap

PCF $\parallel$ ?
+parallelism
+sequentiality
?  
+parallelism
+state
+parallel innocence

IPA ?
+parallelism
+sequentiality

PCF ?
+state
+parallel innocence

IA ?
+parallelism
+state
+parallel innocence
Non-alternating game semantics for IPA

Theorem

*The model GM of games and well-bracketed non-alternating strategies is fully abstract for IPA.*

Definition

An *(non-alternating) play* on game $A$ is a finite non-repetitive sequence of events $a_1 \ldots a_n$ such that for all $1 \leq i \leq n$,

$$\{a_1, \ldots, a_i\} \in C(A).$$

We write $\text{Plays}(A)$ the set of (non-alternating) plays on $A$.

Definition

A *non-alternating strategy* $\sigma : A$ is a subset

$$\sigma \subseteq \text{Plays}(A)$$

satisfying some conditions.

---

Non-alternating plays

A term:

$$\lambda f : U \to U. \text{newref } r \text{ in } f (r := \text{tt}); !r : (U \to U) \to B$$

A play

$$(U \to U) \to B$$

A game

$$q^- \quad q^+ \quad \text{tt}^+ \quad \text{ff}^+

q^- \quad (\)^- 

(\)^+$$
Non-alternating plays

A term:

$$\lambda f : U \to U \cdot \text{newref } r \text{ in } f \ (r := \texttt{tt}); \ !r : (U \to U) \to B$$

A play

$$(U \to U) \to B$$

A game

$$q^- \quad q^+ \quad \texttt{tt}^+ \quad \texttt{ff}^+$$

$$q^- \quad (^-)$$

$$()^+$$
Non-alternating plays

A term:

\[ \lambda f : U \to U . \text{newref } r \text{ in } f \ (r := \texttt{tt}); \ !r \ : \ (U \to U) \to \mathbb{B} \]

A play

\[(U \to U) \to \mathbb{B}\]

A game

\[
\begin{align*}
q^- & \quad q^+ \\
\text{tt}^+ & \quad \text{ff}^+ \\
(\) & \quad ()^+
\end{align*}
\]
Non-alternating plays

A term:

\[ \lambda f^{U \rightarrow U} \cdot \text{newref } r \text{ in } f (r := \texttt{tt}); !r : (U \rightarrow U) \rightarrow B \]

A play

A game
Non-alternating plays

A term:

\[ \lambda f^{U \to U}. \text{newref } r \in f \ (r := \text{tt}); \ !r \ : \ (U \to U) \to B \]

A play

\[(U \to U) \to B\]

A game

\[q^- \quad q^+ \quad (\text{tt}^+ \quad \text{ff}^+) \quad (\text{tt}^- \quad \text{ff}^-) \quad (\text{tt}^+ \quad \text{ff}^-) \quad (\text{tt}^- \quad \text{ff}^+) \]
Non-alternating plays

A term:

$$\lambda f : U \to U . \text{newref } r \text{ in } f \ (r := \text{tt}); !r \ : \ (U \to U) \to B$$

A play

$$(U \to U) \to B$$

A game

$$q^- \quad q^+$$

$$q^- \quad ()^-$$

$$q^- \quad (())^+$$

$$q^- \quad (tt^+ \quad ff^+)$$
Non-alternating plays

A term:

\[ \lambda f^{U \rightarrow U}. \text{newref } r \text{ in } f (r := \text{tt}); !r : (U \rightarrow U) \rightarrow B \]

A play

\[(U \rightarrow U) \rightarrow B\]

A game

\[\text{q}^- \quad \text{q}^+ \quad \text{tt}^+ \quad \text{ff}^+ \]

\[\text{q}^- \quad \text{q}^+ \quad \text{tt}^+ \quad \text{ff}^+ \]

\[\text{q}^- \quad \text{q}^+ \quad \text{tt}^+ \quad \text{ff}^+ \]
Non-alternating plays

A term:

$$\lambda f : U \rightarrow U. \text{newref } r \text{ in } f \ (r := \texttt{tt}); \ !r \ : \ (U \rightarrow U) \rightarrow \mathbb{B}$$

A play

$$(U \rightarrow U) \rightarrow \mathbb{B}$$

A game

$$q^- \quad q^+ \quad \texttt{tt}^+ \quad \texttt{ff}^+$$

$$q^- \quad (\_)^- \quad \texttt{tt} \quad \texttt{ff}$$

$$\_ \quad \_ \quad \_ \quad \_+$$
Non-alternating plays

A term:

$$\lambda f : U \rightarrow U. \text{newref } r \text{ in } f(r := \text{tt}); \text{!r} : (U \rightarrow U) \rightarrow B$$
Non-alternating plays

A term:

$$\lambda f : U \rightarrow U . \text{newref } r \text{ in } f (r := \text{tt}); !r : (U \rightarrow U) \rightarrow \mathbb{B}$$

A play

$$\begin{align*}
( & U \rightarrow U ) \rightarrow \mathbb{B} \\
& q^- \\
& \phantom{+} q^+ \\
& q^- \\
& \phantom{+} ()^- \\
& ()^+ \\
& \phantom{+} ff^+
\end{align*}$$

A game

$$\begin{align*}
q^- \\
\phantom{+} q^+ \\
\phantom{+} \text{tt}^+ \\
\phantom{-} \text{ff}^+ \\
q^- \\
\phantom{+} ()^- \\
\phantom{+} ()^+
\end{align*}$$
Non-alternating plays

A term:

\[ \lambda f : \mathbb{U} \rightarrow \mathbb{U}. \text{newref } r \text{ in } f \ (r := \texttt{tt}); !r \ : \ (\mathbb{U} \rightarrow \mathbb{U}) \rightarrow \mathbb{B} \]
Non-alternating plays

A term:

\[ \lambda f^{U \to U}. \text{newref } r \text{ in } f (r := \text{tt}); !r : (U \to U) \to B \]
Non-alternating plays

A term:

\[ \lambda f : U \rightarrow U. \text{newref } r \text{ in } f (r := \text{tt}); !r : (U \rightarrow U) \rightarrow \mathbb{B} \]

A play

\[
(\rightarrow U) \rightarrow \mathbb{B}
\]

\[ q^- \]

\[ q^+ \]

\[ ()^+ \]

\[ ()^- \]

\[ \text{tt}^+ \]

A game

\[ q^- \]

\[ q^+ \]

\[ \text{tt}^+ \]

\[ \text{ff}^+ \]

\[ ()^- \]

\[ ()^+ \]
Question: can a program without state realize these two plays?

\[
(B \rightarrow U) \rightarrow U
\]

\[
(B \rightarrow U) \rightarrow U
\]
Question: can a program without state realize these two plays?

\[
\lambda f^{\mathbb{B} \to \mathbb{U}}. \text{let } \begin{pmatrix} x &= f \text{tt} \\ y &= f \text{ff} \end{pmatrix} \text{ in } x; y : (\mathbb{B} \to \mathbb{U}) \to \mathbb{U}
\]
Question: can a program without state realize these two plays?

\[
\lambda f^{\mathbb{B} \to U}. \text{let } \begin{pmatrix} x &= f^{tt} \\ y &= f^{ff} \end{pmatrix} \text{ in } x; \ y : (\mathbb{B} \to U) \to U
\]
Question: can a program without state realize these two plays?

\[
\lambda f \mathbb{B} \to U. \text{let } \begin{cases} x = f \mathbf{tt} \\ y = f \mathbf{ff} \end{cases} \text{ in } x; y : (\mathbb{B} \to U) \to U
\]

\[\leftrightarrow \text{ concurrent games}^{3}\]

---

3 Family of models initiated by Abramsky and Melliès (1999), then Melliès, Mimram, Faggian, Piccolo (2000s), then Rideau, Winskel, Castellan, C., Paquet, Alcolei, de Visme etc... (2010s).
Roadmap

PCF II → +state → IPA

+parallel innocence

→ +state

PCF → +parallel innocence → IA

+sequentiality

CG

+parallelism

+sequentiality

+parallelism
Partially ordered plays: augmented configurations

**Definition**

An augmentation on $A$ is a conflict-free event structure $q = \langle |q|, \leq_q \rangle$ where

$$C(q) \subseteq C(A).$$

An augmentation

\[
\begin{array}{c}
(B \rightarrow U) \rightarrow U \\
\downarrow \\
q^{-} \\
\downarrow \\
q^{-} \\
\downarrow \\
q^{+} \\
\downarrow \\
(\_)^{-} \\
\downarrow \\
(\_)^{+} \\
\end{array}
\]

(\rightarrow is the immediate causality relation).
Definition

An **augmentation** on $A$ is a conflict-free event structure $q = \langle |q|, \leq_q \rangle$ where

$$C(q) \subseteq C(A).$$

An augmentation

$$(\mathbb{B} \rightarrow \mathbb{U}) \rightarrow \mathbb{U}$$

(→ is the **immediate causality relation**).
Question: is this augmentation realizable?
Question: is this augmentation realizable?

\[ \lambda f^{B \rightarrow U}. f \text{ tt} : (B \rightarrow U) \rightarrow U \]
Question: is this augmentation realizable?

\[(\mathcal{B} \rightarrow \mathcal{U}) \rightarrow \mathcal{U}\]
Question: is this augmentation realizable?

\[(B \rightarrow U) \rightarrow U\]

Definition

An augmentation \( q \) on \( A \) is **courteous** iff for all \( a_1 \rightarrow_q a_2 \) such that \( \neg(a_1 \rightarrow_A a_2) \), we have \( \text{pol}_A(a_1) = - \) and \( \text{pol}_A(a_2) = + \).

We write \( \text{Aug}(A) \) for the set of **courteous augmentations** on \( A \).
Question: is this augmentation realizable?

\[(U \rightarrow U) \rightarrow B\]
Question: is this augmentation realizable?

\[ \lambda f^{\mathbb{U} \to \mathbb{U}}. \text{newref } r \text{ in } f \ (r := \texttt{tt}); \ !r : \ (\mathbb{U} \to \mathbb{U}) \to \mathbb{B} \]
Question: is this augmentation realizable?

\[
\lambda f^{\text{U} \to \text{U}}. \text{newref } r \text{ in } f (r := \text{tt}); \! r : (\text{U} \to \text{U}) \to \text{B}
\]
Question: is this augmentation realizable?

Definition

A (concurrent) strategy \( \sigma : A \) is a **non-empty, prefix-closed** subset

\[ \sigma \subseteq \text{Aug}(A) \]

closed under extensions by Opponent events.
Theorem

The model $\text{CG}$ of games and (well-bracketed) concurrent strategies is intensionally fully abstract for IPA.

Proof.

If $\sigma : A$ is a strategy, then

$$\text{Plays}(\sigma) = \bigcup \{ \text{Plays}(q) \mid q \in \sigma \}$$

is a strategy in the Ghica-Murawski sense.

This forms a functor

$$\text{Plays}(-) : \text{CG} \to \text{GM}$$

preserving the interpretation.

\[ \]
Roadmap

\[ \text{PCF} \parallel \text{CG}_{\text{inn}} \xrightarrow{+\text{state}} \xleftarrow{+\text{parallel innocence}} \text{IPA} \xrightarrow{+\text{parallelism}} \text{CG} \]

\[ +\text{parallelism} \quad +\text{sequentiality} \quad +\text{parallelism} \quad +\text{sequentiality} \]

\[ \text{PCF} \xrightarrow{+\text{state}} \xleftarrow{+\text{parallel innocence}} \text{IA} \]

\[ ? \quad \quad \quad \quad \quad \quad ? \]
Question: which of these two is realizable only with state?

\[
\begin{align*}
\mathbb{U} & \rightarrow \mathbb{U} \rightarrow \mathbb{U} \\
(\mathbb{U} \rightarrow \mathbb{U} \rightarrow \mathbb{U}) & \rightarrow \mathbb{U}
\end{align*}
\]
Question: which of these two is realizable only with state?

\[
U \rightarrow U \rightarrow U
\]

\[
(U \rightarrow U \rightarrow U) \rightarrow U
\]

**Definition**

An augmentation \( q \in \text{Aug}(A) \) is **innocent** if it has no pattern of the form

\[
\begin{align*}
\triangledown m^- & \rightarrow\cdots \rightarrow m^- \\
\triangle m^+ & \rightarrow m^- \\
\triangle m^- & \rightarrow\cdots \rightarrow m^- \\
\end{align*}
\]

A strategy \( \sigma : A \) is **innocent** if any \( q \in \sigma \) is.
The causal shape of parallel innocence
The causal shape of parallel innocence
The causal shape of parallel innocence
The causal shape of parallel innocence
The causal shape of parallel innocence
The causal shape of parallel innocence
Question: is the following augmentation realizable without state?

\[(\mathbb{U} \rightarrow \mathbb{U}) \rightarrow (\mathbb{U} \rightarrow \mathbb{U}) \rightarrow \mathbb{U}\]
Question: is the following augmentation realizable without state?

\[( \mathbb{U} \rightarrow \mathbb{U} ) \rightarrow ( \mathbb{U} \rightarrow \mathbb{U} ) \rightarrow \mathbb{U} \]
Question: is the following augmentation realizable without state?

\[(\mathbb{U} \rightarrow \mathbb{U}) \rightarrow (\mathbb{U} \rightarrow \mathbb{U}) \rightarrow \mathbb{U}\]
Question: is the following augmentation realizable without state?

\[ (U \rightarrow U) \rightarrow (U \rightarrow U) \rightarrow U \]
Question: is the following augmentation realizable without state?

\[(\mathcal{U} \rightarrow \mathcal{U}) \rightarrow (\mathcal{U} \rightarrow \mathcal{U}) \rightarrow \mathcal{U}\]
Question: is the following augmentation realizable without state?

\[(U \rightarrow U) \rightarrow (U \rightarrow U) \rightarrow U\]

**Definition**

A **grounded causal chain (gcc)** of augmentation \(q \in \text{Aug}(A)\) is

\[\rho = \rho_1 \rightarrow_q \rho_2 \rightarrow_q \ldots \rightarrow_q \rho_n\]

where \(\rho_1\) is minimal in \(q\).
Question: is the following augmentation realizable without state?

\[(U \rightarrow U) \rightarrow (U \rightarrow U) \rightarrow U\]

**Definition**

A grounded causal chain (gcc) of augmentation \( q \in \text{Aug}(A) \) is

\[\rho = \rho_1 \xrightarrow{q} \rho_2 \xrightarrow{q} \cdots \xrightarrow{q} \rho_n\]

where \( \rho_1 \) is minimal in \( q \).

**Definition**

A strategy \( \sigma : A \) is **visible** iff for all \( \rho \in \text{gcc}(\sigma), \rho \in C(A) \).
Full abstraction for $\text{PCF}_{\parallel}^5$

**Theorem**

The model $\text{CG}_{\text{inn}}$ of games and deterministic, (visible) parallel innocent strategies is intensionally fully abstract for $\text{PCF}_{\parallel}$.

**Proof.**

Via finite definability up to observational equivalence.

---

Full abstraction for $\text{PCF}_\parallel$ \(^5\)

Theorem

The model $\text{CG}_{\text{inn}}$ of games and deterministic, (visible) parallel innocent strategies is intensionally fully abstract for $\text{PCF}_\parallel$.

Proof.

Via finite definability up to observational equivalence.

---

Sequentiality and full abstraction for $\text{IA}^6$

**Theorem**

The model $\text{CG}_{\text{seq}}$ of games and deterministic sequential strategies is intensionally fully abstract for $\text{IA}$.

**Proof.**

If $\sigma : A$ is well-bracketed sequential deterministic, then

$$\text{AltPlays}(\sigma) = \bigcup \{ \text{AltPlays}(q) \mid q \in \sigma \}$$

is a strategy in the sense of Abramsky-McCusker. This forms a functor

$$\text{AltPlays}(-) : \text{CG}_{\text{seq}} \rightarrow \text{AM}$$

preserving the interpretation.

---

Wrapping up
Wrapping up

PCF ||
CG_{inn} + parallel innocence + determinism + state

IPA
CG + parallelism + sequentiality + determinism

PCF
CG_{seq} + parallelism + sequentiality + determinism

IA
CG_{inn} + parallel innocence + state

?
Wrapping up

- \( \text{PCF}_{\parallel} \) → \( \text{IPA} \) with state
- \( \text{CG}_{\text{inn}} \) → \( \text{CG} \) with parallel innocence and determinism
- \( \text{PCF} \) → \( \text{IA} \) with state
- \( \text{CG}_{\text{seq,inn}} \) → \( \text{CG}_{\text{seq}} \) with parallel innocence
- \( \text{PCF} \) → \( \text{IPA} \) with sequentiality
- \( \text{CG}_{\text{inn}} \) → \( \text{CG} \) with parallelism

+parallelism
+sequentiality
+parallelism
+sequentiality
+parallelism
+sequentiality
+parallelism
+sequentiality
+parallelism
+sequentiality
+parallelism
+sequentiality
+parallelism
+sequentiality
Wrapping up

\[
\begin{array}{ccc}
\text{PCF}_{||} & \xrightarrow{+\text{state}} & \text{IPA} \\
\text{CG}_{\text{inn}} & \xleftarrow{+\text{parallel innocence}} & \text{CG} \\
\text{PCF} & \xleftarrow{+\text{sequentiality}} & \text{IA} \\
\text{CG}_{\text{seq,inn}} & \xrightarrow{+\text{parallel innocence}} & \text{CG}_{\text{seq}} \\
& \xleftarrow{+\text{sequentiality}} & \\
\text{HO-Inn} & \xrightarrow{+\text{determinism}} & \\
\end{array}
\]
Wrapping up
Wrapping up
Wrapping up

PCF$_{\parallel}$ \(\xrightarrow{CG_{inn}}\) IPA

PCF \(\xrightarrow{CG_{seq,inn}}\) PPCF \(\xrightarrow{+parallel\ innocence}\) IA \(\xleftarrow{CG_{seq}}\)

PCF \(\xrightarrow{+parallelism}\) IPA

PCF \(\xrightarrow{+sequentiality}\) PPCF

PCF \(\xrightarrow{+probabilities}\) PPCF

PCF \(\xrightarrow{+state}\) IA
Wrapping up
Wrapping up
III. The sequential face
Probabilistic IA

Types.

\[ A, B ::= \mathbb{U} | \mathbb{B} | A \rightarrow B \]

Terms.

\[ M, N ::= x | M \cdot N | \lambda x. M | Y | tt | ff | if M N_1 N_2 | skip | M; N | newref \nu := b \text{ in } M | M := N | !M | \text{rand}() | +\text{state} +\text{probabilities} \]
Roadmap

PCF \rightarrow \text{PPCF} \rightarrow \text{PIA}

\text{CG}_{\text{seq,inn}} \rightarrow \text{PPCF} \leftarrow \text{IA} \rightarrow \text{CG}_{\text{seq}}

+probabilities

+state

+parallel innocence
<table>
<thead>
<tr>
<th>Definiton</th>
<th>A probabilistic strategy $\sigma : A$ is a function $\sigma : \text{Aug}(A) \rightarrow [0, 1]$ satisfying some conditions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conjecture</td>
<td>The category $\text{PCG}$ of games and (well-bracketed) sequential probabilistic strategies is intensionally fully abstract for PIA.</td>
</tr>
<tr>
<td>Proof.</td>
<td>If $\sigma : A$ is a probabilistic concurrent strategy, then setting $\text{AltPlays}(\sigma) : \text{AltPlays}(A) \rightarrow [0, 1]$ $s \mapsto \sum_{q \in \sigma \text{ and } s \in \text{AltPlays}(q)} \sigma(q)$ yields a probabilistic strategy in the sense of Danos-Harmer. This induces $\text{AltPlays}(\cdot) : \text{PCG} \rightarrow \text{DH}$.</td>
</tr>
</tbody>
</table>

\[ \text{AltPlays}(\sigma)(s) = \sum_{q \in \sigma \text{ and } s \in \text{AltPlays}(q)} \sigma(q) \]
Roadmap
Roadmap

PCF → PPCF
+ confidence
PCG_{seq,inn} → + parallel innocence

+ state

PCF ← PPCF
+ confidence
PCG_{seq,inn} ← + parallel innocence

IA
CG_{seq} → + confidence

AM
CG_{seq} ← + parallel innocence

HO-Inn

PIA
PCG_{seq} ← + parallel innocence

IV. Conclusions
Summary of the talk
Summary of the talk

+parallelism
+probabilities
+state
Summary of the talk
Summary of the talk

PCF∥ → PPCF∥ → PPCF → IPA → PIA

+parallelism +probabilities +state
Perspectives

A. Probabilistic Relational Collapse
Refresher on the relational model

Theorem

The category $\text{Rel}$ has **sets as objects and relations**

$$R \subseteq A \times B$$

as morphisms from $A$ to $B$.

*It is a compact closed category with biproducts.*
Refresher on the relational model

Theorem

The category $\text{Rel}$ has sets as objects and relations

$$R \subseteq A \times B$$

as morphisms from $A$ to $B$.  

It is a compact closed category with biproducts.

$$\text{L}B \text{M} = \{tt, ff\}$$
Refresher on the relational model

Theorem

The category $\text{Rel}$ has sets as objects and relations

$$R \subseteq A \times B$$

as morphisms from $A$ to $B$.

*It is a compact closed category with biproducts.*

\[
\begin{align*}
(B) &= \{\text{tt, ff}\} \\
(U) &= \{(())\}
\end{align*}
\]
Refresher on the relational model

Theorem

The category $\textbf{Rel}$ has **sets** as objects and **relations**

$$R \subseteq A \times B$$

as morphisms from $A$ to $B$.

*It is a compact closed category with biproducts.*

\[
\begin{align*}
\langle \mathbf{B} \rangle &= \{ \text{tt, ff} \} \\
\langle \mathbf{U} \rangle &= \{ () \} \\
\langle A \to B \rangle &= \end{align*}
\]
Refresher on the relational model

**Theorem**

The category \( \text{Rel} \) has **sets** as objects and **relations**

\[
R \subseteq A \times B
\]

as morphisms from \( A \) to \( B \).

It is a **compact closed category with biproducts**.

\[
\begin{align*}
\langle B \rangle &= \{ \texttt{tt}, \texttt{ff} \} \\
\langle U \rangle &= \{ () \} \\
\langle A \to B \rangle &= \langle A \rangle^* \otimes \langle B \rangle
\end{align*}
\]
Refresher on the relational model

Theorem

The category \textbf{Rel} has \textit{sets} as objects and \textit{relations}

\[ R \subseteq A \times B \]

as morphisms from \(A\) to \(B\).

It is a \textbf{compact closed} category with \textbf{biproducts}.

\[
\begin{align*}
\langle B \rangle &= \{ \text{tt, ff} \} \\
\langle U \rangle &= \{ () \} \\
\langle A \rightarrow B \rangle &= \langle A \rangle \otimes \langle B \rangle
\end{align*}
\]
Refresher on the relational model

Theorem

The category $\mathbf{Rel}$ has sets as objects and relations

$$R \subseteq A \times B$$

as morphisms from $A$ to $B$.

It is a compact closed category with biproducts.

$$(|B|) = \{\text{tt, ff}\}$$

$$(|U|) = \{()\}$$

$$(|A \rightarrow B|) = (|A|) \times (|B|)$$
Theorem

The category $\textbf{Rel}$ has sets as objects and relations

$$R \subseteq A \times B$$

as morphisms from $A$ to $B$.

It is a compact closed category with biproducts.

$$(\bot) = \{\text{tt}, \text{ff}\}$$

$$(\bot) = \emptyset$$

$$(A \rightarrow B) = (|A|) \times (|B|)$$

$$(!A) = \mathcal{M}_f(|A|)$$
Refresher on the relational model

Theorem

The category Rel has sets as objects and relations

\[ R \subseteq A \times B \]

as morphisms from A to B.

It is a compact closed category with biproducts.

\[
\begin{align*}
(B) &= \{tt, ff\} \\
(U) &= \{()\} \\
(A \rightarrow B) &= (A) \times (B)
\end{align*}
\]
Refresher on the relational model

Theorem

The category $\text{Rel}$ has sets as objects and relations

$$R \subseteq A \times B$$

as morphisms from $A$ to $B$.

It is a compact closed category with biproducts.

$$(|B|) = \{tt, ff\}$$

$$(|U|) = \{(())\}$$

$$(|A \to B|) = (|A| + 1) \times (|B|)$$
Refresher on the relational model

**Theorem**

*The category* \( \text{Rel} \) *has sets as objects and relations*

\[ R \subseteq A \times B \]

*as morphisms from* \( A \) *to* \( B \).

*It is a compact closed category with biproducts.*

\[
\begin{align*}
(\|B\|) &= \{ \texttt{tt}, \texttt{ff} \} \\
(\|U\|) &= \{ () \} \\
(\mathbb{A} \rightarrow B) &= ((\|A\| + 1) \times \|B\|) \\
\lambda f^{\mathbb{B} \rightarrow U} \cdot f \texttt{tt} &= \begin{cases} 
(\mathbb{B} \rightarrow U) \rightarrow U \\
((\texttt{tt}, ()), ()) \\
((\star, ()), ()) 
\end{cases}
\end{align*}
\]
Refresher on the relational model

**Theorem**

The category \textbf{Rel} has sets as objects and relations

\[ R \subseteq A \times B \]

as morphisms from \( A \) to \( B \).

It is a compact closed category with biproducts.

\[ (|B|) = \{ \texttt{tt}, \texttt{ff} \} \]

\[ (|U|) = \{ () \} \]

\[ (|A \rightarrow B|) = (|A| + 1) \times (|B|) \]

\[ \lambda f^{B \rightarrow U} \cdot f \texttt{tt} = \left\{ \begin{array}{l}
(B \rightarrow U) \rightarrow U \\
((\texttt{tt}, ()), ()) \\
((\star, ()), ())
\end{array} \right\} \]

\[ (x, z) \in R_2 \circ R_1 \iff \exists y, (x, y) \in R_1 \& (y, z) \in R_2 \]
Types as games

\[ [U] = \begin{array} {c}
q^-\\
| \\
( )^+
\end{array} \]

\[ [B] = \begin{array} {c}
q^- \\
| \\
\circ \circ \circ \circ \circ \\
\circ \circ \\
tt^+ \\
ff^+
\end{array} \]
Types as games

\[
[U] = q^{-} \\
\downarrow \\
()^{+}
\]

\[
[B] = q^{-} \\
\downarrow \\
\begin{array}{c}
\text{tt}^{+} \\
\hline
\hline
\text{ff}^{+}
\end{array}
\]

**Definition**

If \( B \) has exactly one minimal event;

\[
|A \rightarrow B| = |A| + |B|
\]

\[
\text{pol}_{A \rightarrow B} = [-\text{pol}_A, \text{pol}_B]
\]

\[
\leq_{A \rightarrow B} = \{(a_1, a_2) \mid a_1 \leq_A a_2\} \\
\cup \{(b_1, b_2) \mid b_1 \leq_B b_2\} \\
\cup \{(\min(B), a) \mid a \in |A|\}
\]

\[\lambda A, B = [\lambda A, \lambda B] \]
Types as games

\[
[u] = \begin{array}{c}
q^- \\
()^+
\end{array}
\]

\[
[b] = \begin{array}{c}
q^- \\
\text{tt}^+ \\
\sim \sim \sim 
\text{ff}^+
\end{array}
\]

**Definition**

If \( B \) has exactly one minimal event;

\[
|A \rightarrow B| = |A| + |B|
\]

\[
\text{pol}_{A \rightarrow B} = [-\text{pol}_A, \text{pol}_B]
\]

\[
\leq_{A \rightarrow B} = \{(a_1, a_2) \mid a_1 \leq_A a_2\} \\
\cup \{(b_1, b_2) \mid b_1 \leq_B b_2\} \\
\cup \{(\min(B), a) \mid a \in |A|\}
\]

**Example**

\[
[\left(\left(u \rightarrow u\right) \rightarrow b\right) = \begin{array}{c}
q^- \\
q^+ \\
\text{tt}^+ \\
\sim \sim \sim 
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
q^- \\
()^+
\end{array}
\]
Types as games

\[ [U] = q^-, Q \]

\[ [B] = q^- \]

**Definition**

If \( B \) has exactly one minimal event;

\[ |A \rightarrow B| = |A| + |B| \]

\[ \text{pol}_{A \rightarrow B} = [-\text{pol}_A, \text{pol}_B] \]

\[ \leq_{A \rightarrow B} = \{(a_1, a_2) \mid a_1 \leq_A a_2\} \]

\[ \cup \{(b_1, b_2) \mid b_1 \leq_B b_2\} \]

\[ \cup \{(\min(B), a) \mid a \in |A|\} \]

**Example**

\[ [(U \rightarrow U) \rightarrow B] = \]

\[ q^- \]

\[ q^+ \]

\[ \text{tt}^+ \]

\[ \sim \text{ff}^+ \]

\[ q^- \]

\[ ()^- \]

\[ ()^+ \]
Types as games

\[ [U] = q^-, Q \]

\[ (+) +, A \]

Definition

If \( B \) has exactly one minimal event;

\[ |A \rightarrow B| = |A| + |B| \]

\[ \text{pol}_{A \rightarrow B} = [-\text{pol}_A, \text{pol}_B] \]

\[ \leq_{A \rightarrow B} = \{(a_1, a_2) \mid a_1 \leq_A a_2\} \]

\[ \cup \{(b_1, b_2) \mid b_1 \leq_B b_2\} \]

\[ \cup \{(\text{min}(B), a) \mid a \in |A|\} \]

Example

\[ [(U \rightarrow U) \rightarrow B] = \]

\[ q^- \]

\[ \text{tt}^+ \]

\[ \sim \text{ff}^+ \]

\[ q^- \]

\[ (\cdot)^- \]

\[ (\cdot)^+ \]
Types as games

\[
[U] = q^{-}, Q
\]

\[
(B) = \begin{array}{c}
q^{-}, Q \\
\text{tt}^+, \mathcal{A} \\
\sim \\
\text{ff}^+, \mathcal{A}
\end{array}
\]

**Definition**

If \( B \) has exactly one minimal event;

\[
|A \rightarrow B| = |A| + |B|
\]

\[
\text{pol}_{A \rightarrow B} = [-\text{pol}_{A}, \text{pol}_{B}]
\]

\[
\leq_{A \rightarrow B} = \{(a_1, a_2) \mid a_1 \leq_{A} a_2\}
\]

\[
\cup \{(b_1, b_2) \mid b_1 \leq_{B} b_2\}
\]

\[
\cup \{\text{min}(B), a\} \mid a \in |A|
\]

**Example**

\[
[(U \rightarrow U) \rightarrow B] =
\]

\[
\begin{array}{c}
q^{-} \\
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
q^{-} \quad ( )^- \\
\quad ( )^+
\]

\[
\begin{array}{c}
q^+ \\
\quad \\
\quad \\
\quad
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\quad \\
\quad \\
\quad
\end{array}
\]

\[
\begin{array}{c}
\text{ff}^+ \\
\quad \\
\quad \\
\quad
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
q^- \\
\quad \\
\quad \\
\quad
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^+ \\
\sim \\
\text{ff}^+
\end{array}
\]
Types as games

\[ [U] = q^{-, Q} \]

\[ (\,)^{+, A} \]

\[ [B] = q^{-, Q} \]

\[ \text{tt}^{+, A} \sim \text{ff}^{+, A} \]

**Definition**

If \( B \) has exactly one minimal event;

\[ |A \rightarrow B| = |A| + |B| \]

\[ \text{pol}_{A \rightarrow B} = [\neg \text{pol}_A, \text{pol}_B] \]

\[ \leq_{A \rightarrow B} = \{(a_1, a_2) \mid a_1 \leq_A a_2\} \]

\[ \bigcup \{(b_1, b_2) \mid b_1 \leq_B b_2\} \]

\[ \bigcup \{(\min(B), a) \mid a \in |A|\} \]

\[ \lambda_{A, B} = [\lambda_A, \lambda_B] \]

**Example**

\[ [(U \rightarrow U) \rightarrow B] = \]

\[ q^{-} \]

\[ \text{tt}^{+} \sim \text{ff}^{+} \]

\[ q^{-} \]

\[ (\,)^{-} \]

\[ (\,)^{+} \]
Types as games

\[
[\emptyset] = \begin{array}{c}
q^{-}, Q \\
\end{array}
\]

\[
(U) = \begin{array}{c}
q^{-}, Q \\
\end{array}
\]

\[
[B] = \begin{array}{c}
q^{-}, Q \\
\end{array}
\]

\[
\lambda_{A, B} = [\lambda_A, \lambda_B]
\]

\[
\lambda_{A, B} = [-\lambda_A, \lambda_B]
\]

\[
\lambda_{A, B} = [\lambda_A, \lambda_B]
\]

\[
\lambda_{A, B} = [\lambda_A, \lambda_B]
\]

Definition

If \( B \) has exactly one minimal event;

\[
|A \to B| = |A| + |B|
\]

\[
\text{pol}_{A \to B} = [-\text{pol}_A, \text{pol}_B]
\]

\[
\leq_{A \to B} = \{(a_1, a_2) | a_1 \leq_A a_2\}
\]

\[
\cup \{(b_1, b_2) | b_1 \leq_B b_2\}
\]

\[
\cup \{(\text{min}(B), a) | a \in |A|\}
\]

\[
\lambda_{A, B} = [\lambda_A, \lambda_B]
\]

Example

\[
[(\emptyset \to \emptyset) \to B] =
\]

\[
\begin{array}{c}
q^{-}, Q \\
\end{array}
\]

\[
\begin{array}{c}
\text{tt}^{+, A} \sim \text{ff}^{+, A}
\end{array}
\]

\[
\begin{array}{c}
q^{-}, Q \\
\end{array}
\]

\[
\lambda_{A, B} = [\lambda_A, \lambda_B]
\]

\[
\lambda_{A, B} = [\lambda_A, \lambda_B]
\]

\[
\lambda_{A, B} = [\lambda_A, \lambda_B]
\]
Types as games

\[ [\mathbb{U}] = q^{-}, Q \]
\[ (\mathbf{)}^{+}, A \]

\[ [\mathbb{B}] = q^{-}, Q \]
\[ \text{tt}^{+}, A \sim \text{ff}^{+}, A \]

**Definition**

If \( B \) has exactly one minimal event;

\[ |A \rightarrow B| = |A| + |B| \]

\[ \text{pol}_{A \rightarrow B} = [-\text{pol}_A, \text{pol}_B] \]

\[ \leq_{A \rightarrow B} = \{(a_1, a_2) \mid a_1 \leq_A a_2\} \]
\[ \cup \{(b_1, b_2) \mid b_1 \leq_B b_2\} \]
\[ \cup \{(\text{min}(B), a) \mid a \in |A|\} \]

\[ \lambda_{A, B} = [\lambda_A, \lambda_B] \]

**Example**

\[ [(\mathbb{U} \rightarrow \mathbb{U}) \rightarrow \mathbb{B}] = \]
\[ q^{-}, Q \]
\[ q^{+}, Q \]
\[ \text{tt}^{+}, A \sim \text{ff}^{+}, A \]

\[ q^{-}, Q \]
\[ \text{tt}^{+}, A \sim \text{ff}^{+}, A \]

\[ (\mathbf{)}^{-}, A \]

\[ (\mathbf{)}^{+}, A \]

**Definition**

A configuration \( x \in C(A) \) is **complete** iff every question has an answer. Write \( \int A \) the set of non-empty complete configurations of \( A \).
Games and the web

Theorem

For any type $A$,

$$\int [A] \cong (A)$$
Games and the web

Theorem

*For any type \( A \),*

\[
\int[A] \cong (A)
\]
Games and the web

Theorem

For any type $A$,

\[ \int [A] \cong (A) \]
Collapse of strategies

If $\sigma : A$ is a strategy, write $C(\sigma) = \bigcup \{ C(q) \mid q \in \sigma \}$.

Definition

$\int \sigma = C(\sigma) \cap (\int A)$

Example

$\int \left( \left( (U \rightarrow U) \rightarrow B \right), \left( (U \rightarrow U) \rightarrow B \right) \right) \sim \left\{ \left( (U \rightarrow U) \rightarrow B \right), \left( (((), (), ()), \text{tt} \right), \left( (((), (), ()), \text{ff} \right) \right\}$
Composition of strategies

**Definition**

$q \in \text{Aug}(A \rightarrow B)$ and $p \in \text{Aug}(B \rightarrow C)$ are **causally compatible** iff

\begin{align*}
(1) & \quad |q| = x_A + x_B \quad \& \quad |p| = x_B + x_C \\
(2) & \quad \leq_q \cup \leq_p \quad \text{is acyclic}.
\end{align*}

Then, their **interaction** is

$$p \otimes q = (x_A + x_B + x_C, (\leq_q \cup \leq_p)^*)$$

Their **composition** is

$$p \odot q = p \otimes q \upharpoonright A \rightarrow C$$

**Definition**

If $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$ are strategies, then their **composition** is

$$\tau \odot \sigma = \{ p \odot q \mid q \in \sigma \text{ and } p \in \tau \text{ are causally compatible}\}$$
Example of composition

Overall composition

\[
\begin{pmatrix}
(U \to U) \to B \\
(U \to U) \to B
\end{pmatrix}
\quad ,
\quad
\begin{pmatrix}
(U \to U) \\
(U \to U)
\end{pmatrix}
\quad \star
\quad
\begin{pmatrix}
(U \to U)
\end{pmatrix}
\quad =
\]
Example of composition

Overall composition

\[
\left( (U \to U) \to B \right), \quad \left( (U \to U) \to B \right) \quad \circ \quad \left( U \to U \right) =
\]

\[
\left( (U \to U) \to B \right), \quad \left( (U \to U) \to B \right) \quad \circ \quad \left( U \to U \right) =
\]
Example of composition

Overall composition

\[
\begin{pmatrix}
(U \to U) \to B
\end{pmatrix},
\begin{pmatrix}
(U \to U) \to B
\end{pmatrix}
\] \quad \circ \quad \begin{pmatrix}
U \to U
\end{pmatrix}
Example of composition

Overall composition

\[
\left( (U \to U) \to B \right) \circ \left( (U \to U) \to B \right) =
\]

\[
\left( U \to U \right)
\]
Example of composition

Overall composition

\[
\left( (U \rightarrow U) \rightarrow B \right) \cdot \left( (U \rightarrow U) \rightarrow B \right) \cdot \left( U \rightarrow U \right) =
\]

\[
\left( (U \rightarrow U) \rightarrow B \right) \cdot \left( (U \rightarrow U) \rightarrow B \right)
\]

\[
\left( (U \rightarrow U) \rightarrow B \right) \cdot \left( U \rightarrow U \right) =
\]

\[
\left( (U \rightarrow U) \rightarrow B \right) \cdot \left( U \rightarrow U \right)
\]
Example of composition

Overall composition

\[
\begin{pmatrix}
(U \rightarrow U) \rightarrow B
\end{pmatrix}
\begin{pmatrix}
(U \rightarrow U) \rightarrow B
\end{pmatrix}
\begin{pmatrix}
(U \rightarrow U)
\end{pmatrix}
= \\
\begin{pmatrix}
(U \rightarrow U) \rightarrow B
\end{pmatrix}
\begin{pmatrix}
(U \rightarrow U)
\end{pmatrix}
\begin{pmatrix}
(U \rightarrow U) \rightarrow B
\end{pmatrix}
\begin{pmatrix}
(U \rightarrow U)
\end{pmatrix}
\]
Example of composition

Overall composition

\[
\left( \begin{array}{c}
(U \to U) \to B
\end{array} \right) \quad ,
\left( \begin{array}{c}
(U \to U) \to B
\end{array} \right) \quad \circ 
\left( \begin{array}{c}
(U \to U)
\end{array} \right) =
\left( \begin{array}{c}
B
\end{array} \right)
\]

\[
\left( \begin{array}{c}
(U \to U) \to B
\end{array} \right) \quad ,
\left( \begin{array}{c}
(U \to U) \to B
\end{array} \right) \quad \circ 
\left( \begin{array}{c}
(U \to U)
\end{array} \right) =
\left( \begin{array}{c}
B
\end{array} \right)
\]

\[
\left( \begin{array}{c}
(U \to U) \to B
\end{array} \right) \quad ,
\left( \begin{array}{c}
(U \to U) \to B
\end{array} \right) \quad \circ 
\left( \begin{array}{c}
(U \to U)
\end{array} \right) =
\left( \begin{array}{c}
B
\end{array} \right)
\]
Example of composition

Overall composition

\[
\left( (U \rightarrow U) \rightarrow B \right) \circ \left( (U \rightarrow U) \rightarrow B \right) \cong \left( B \rightarrow B \right)
\]

\[
\int \left( (U \rightarrow U) \rightarrow B \right) \circ \left( (U \rightarrow U) \rightarrow B \right) \cong \int \left( B \rightarrow B \right)
\]
Example of composition

Overall composition

\[
\left( (U \rightarrow U) \rightarrow B \right), (U \rightarrow U) \rightarrow B \right) \circ \left( U \rightarrow U \right) = \left( B \right)
\]

\[
\left\{ (U \rightarrow U) \rightarrow B \right\} \circ \int \left( U \rightarrow U \right) = \left( B \right)
\]
Example of composition

Overall composition

\[
\left( (U \to U) \to B \right) \circ \left( (U \to U) \to B \right) = \left( B \right)
\]

\[
\left\{ (U \to U) \to B \left\{ ((()), (()), (tt)) \right\} \circ \left\{ U \to U \left\{ ((()), (()), (ff)) \right\} \right\} =
\]

\[
\left\{ B \right\}
\]
Example of composition

Overall composition

\[
\begin{pmatrix}
    (U \rightarrow U) \rightarrow B \\
    ((()), ()), tt)
\end{pmatrix}
\quad \circ 
\quad \begin{pmatrix}
    U \rightarrow U \\
    (((), ()), ff)
\end{pmatrix}
\quad =
\quad \begin{pmatrix}
    B \\
    tt \\
    ff
\end{pmatrix}
\]
The deadlock-free lemma

Lemma

For $\sigma : A \rightarrow B$, $\tau : B \rightarrow C$ visible strategies, $q \in \sigma$ and $p \in \tau$ such that

\[(1) \quad |q| = x_A + x_B \quad \& \quad |p| = x_B + x_C ,\]

then, $p$ and $q$ satisfy:

\[(2) \quad \leq_q \cup \leq_p \text{ is acyclic} \]

Proof.

By descent on the justification pointers. 

---

\footnote{9P. Baillot, V. Danos, T. Ehrhard, L. Regnier, Timeless games, CSL 1997} \footnote{10P.-A. Melliès, Asynchronous games 4: A fully complete model of propositional linear logic, LICS 2005.} \footnote{11P. Boudes, Thick subtrees, games and experiments, TLCA 2009.}
The deadlock-free lemma

Lemma

For \(\sigma : A \rightarrow B, \tau : B \rightarrow C\) visible strategies, \(q \in \sigma\) and \(p \in \tau\) such that

\[(1) \quad |q| = x_A + x_B \land |p| = x_B + x_C,\]

then, \(p\) and \(q\) satisfy:

\[(2) \quad \leq_q \cup \leq_p\ 	ext{is acyclic}\]

Proof.

By descent on the justification pointers.

Theorem

\(\int(\_): CG_{vis} \rightarrow Rel\)

---

The deadlock-free lemma \(^9\) \(^10\) \(^11\)

**Lemma**

For \(\sigma : A \to B\), \(\tau : B \to C\) visible strategies, \(q \in \sigma\) and \(p \in \tau\) such that

\[(1) \quad |q| = x_A + x_B \quad \text{and} \quad |p| = x_B + x_C ,\]

then, \(p\) and \(q\) satisfy:

\[(2) \quad \leq_q \cup \leq_p \quad \text{is acyclic}\]

**Proof.**

By descent on the justification pointers.

**Theorem**

\(\int(\_ \_ \_ \_ \_ \_\_ \_ \_) : \text{CG}_{\text{inn}} \to \text{Rel}\)


Adding probabilities

PCF \(\rightarrow\) PPCF \(\rightarrow\) PIA

PCG\(_{\text{seq,inn}}\) \(\rightarrow\) IA \(\rightarrow\) DH

PCG\(_{\text{seq}}\) \(\rightarrow\) PRel

CG\(_{\text{seq,inn}}\) \(\rightarrow\) HO-Inn

+probabilities \(\rightarrow\) +parallel innocence
+confidence \(\rightarrow\) +confidence
+state \(\rightarrow\) +parallel innocence
The probabilistic relational model \(^{12}\)

**Definition**

\( \textbf{PRel} \) has **sets** as objects, and as morphisms from \( A \) to \( B \), **matrices**

\[
(\alpha_{a,b})_{(a,b) \in A \times B} \in \mathbb{R}_{+}^{A \times B}
\]

with coefficients in \( \mathbb{R}_{+} \) the completed positive reals.

**Definition**

\[
(\beta \circ \alpha)_{a,c} = \sum_{b \in B} \alpha_{a,b} \cdot \beta_{b,c}
\]

**Theorem (Ehrhard, Tasson, Pagani)**

\( \textbf{PRel} \) is fully abstract for \( \text{PPCF} \).

---

### Theorem

$\text{PCG}_{\text{inn}}$ is intensionally fully abstract for PPCF.

### Proof.

If $\sigma : A \rightarrow B$ and $x_A \in \int A$, $x_B \in \int B$, we define

$$(\int \sigma)_{x_A, x_B} = \sum_{\substack{q \in \sigma \mid |q| = x_A + x_B}} \sigma(q)$$

This yields a functor

$$\int(-) : \text{PCG}_{\text{inn}} \rightarrow \text{PRel}$$

preserving the interpretation.
The sequential face

PCF \( \rightarrow \) PPCF

\( \rightarrow \) PCG\textsubscript{seq,inn} \( \rightarrow \) PIA

+probability

+parallel innocence

+state

+confidence

CG\textsubscript{seq,inn}

PRel

CG\textsubscript{seq}

AM

HO-Inn

DH